Family of curves of differential equations

Construct a differential equation of the family of plane curves defined by an algebraic equation.
Suppose that a family of plane curves is described by the implicit one-parameter equation:

F(x, y, C)=0.

We should differentiate the **F(x, y, C)=0** once by considering y as a function of x and then eliminating the parameters C.

If a family of plane curves is given by the two-parameter equation

$F(x, y, C_1, C_2)=0$,

We should differentiate the **$F(x, y, C_1, C_2)=0$,** twice by considering y as a function of x and then eliminating the parameters C_1 and C_2 from the system of three equations.

The similar rule is applied to the case of n-parametric family of plane curves.

Problem: Derive the differential equation for the family of plane curves defined by the equation $y = x^2 - cx$. **Solution :** We differentiate the implicit equation with respect to x: $$\frac{dy}{dx} = 2x - c$$ $$\Rightarrow c = 2x - \frac{dy}{dx}$$ Write this equation jointly with the original algebraic equation and eliminate the parameter C:	$$y = x^2 - x(2x - \frac{dy}{dx})$$ $$\Rightarrow y = x^2 - 2x^2 + x\frac{dy}{dx}$$ $$\Rightarrow x\frac{dy}{dx} = y + x^2$$ As a result, we obtain the implicit differential equation corresponding to the given family of plane curves.

Home Work:

- **(i)** Derive the differential equation for the family of two-parameter plane curves $y = C_1x^2 + C_2x$
- **(ii)** Find the differential equation of the family of curves $y = e^{2x} + C_2e^{-2x}$
- **(iii)** Form the differential equation of all circles which pass through origin and whose centers lie on y-axis.
- **(iv)** Form the differential equation of all circles which pass through origin and whose centers lie on x-axis.
- **(v)** Derive the differential equation for $y = A\cos\alpha x + B\sin\beta x$, where A and B are arbitrary constants
- **(vi)** Form the DE of which $c(y + c)^2 = x^3$ is the complete integral where c is an arbitrary constants
- **(vii)** Form the differential equation of the following cases

a. $y = cx + c - c^3$ **b.** $y = a\cos(mx + b)$ **c.** $xy = Ae^x + Be^{-x}$ **d.** $y = \sin x + c$	**e.** $y = c\sin^{-1} x$ **f.** $y = Ae^{2x} + Be^{-3x} + Ce^x$ **g.** $xy = Ae^x + Be^{-x} + x^2$ **h.** $y = c(x - c)^2$

Formulations of Differential Equation

A **general solution** to a linear ODE is a solution containing a number of arbitrary variables (equal to the order of the ODE) corresponding to the constants of integration. A **particular solution** is derived from the general solution by setting the constants of integration to values that satisfy the initial value conditions of the problem.

Solve: $\frac{dy}{dx} = x$, for the general solution; for the particular solution with y(0)=1; For $\frac{dy}{dx} = x$, the general solution is $$dy = x\,dx$$ Integrate this we get, $\int y\,dy = \int x\,dx$	$\Rightarrow y = \frac{x^2}{2} + c$ [general solution] The initial value y(0)=1 means $$1 = \frac{0}{2} + c \Rightarrow c = 1$$ The particular solution is therefore $y = \frac{x^2}{2} + 1$

Exercises

1. Find the particular solution of the differential equation which satisfies the given initial condition: $\frac{dy}{dx} = 3x^2 - 1$ with y(o)=4.
2. Find the particular solution of the differential equation which satisfies the given initial condition: $\frac{dy}{dx} = 2\sin x$ with $y\left(\frac{\pi}{2}\right) = \pi$.
3. Find the particular solution of the differential equation which satisfies the given initial condition: $\frac{dy}{dx} = \frac{1}{x^2}$ with y(1)=4.
4. Find the particular solution of the differential equation which satisfies the given initial condition: $\frac{dy}{dx} = -\frac{x}{y}$ with . $y(1) = \sqrt{2}$